5. (THEORY OF) GW DETECTION

GW predicted in 1915ish. But researchers were not sure they were real all the way until the 1960s!
→ In GR we cannot use coordinates to define observables, so can GW be transformed away with a coordinate transformation

FEYNMAN "STICKY BEAD" ARGUMENT (CHAPEL HILL CONFERENCE 1957)

Feynman's gravitational wave detector: It is simply two beads sliding freely (but with a small amount of friction) on a rigid rod. As the wave passes over the rod, atomic forces hold the length of the rod fixed, but the proper distance between the two beads oscillates. Thus, the beads rub against the rod, dissipating heat. -Jin

Amazing book on the history of GW:
TRAVELING AT THE SPEED OF THOUGHT by KENNEFICK

[In current GW research: what is the eccentricity of a binary? We cannot use the shape of the orbit, that's gauge dependent! BOSCHINI, LOUTREL, GEROSA, FUMAGALLI (2025) SHAIKH, VARMA et al (2023)]

Interaction of a GW with test masses

↳ eg the LIGO mirror!

SEC 1.3

GEODESIC EQUATION IN GR $ds^2 = g_{\mu\nu} dx^\mu dx^\nu \to \frac{dx^\nu}{d\lambda} d\lambda$

$$\rightsquigarrow \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}(x) \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \qquad (1.66)$$

GEODESIC DEVIATION Two geodesic $x^\mu(\tau)$, $x^\mu(\tau)+\xi^\mu(\tau)$

$$\rightsquigarrow \frac{d^2\xi^\mu}{d\tau} + 2\Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{d\xi^\rho}{d\tau} + \xi^\sigma \partial_\sigma \Gamma^\mu_{\nu\rho}(x)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \qquad (1.69)$$

TT FRAME ... A somewhat special frame where GW have simple expression ... what's physically?
Take a test mass at rest at $\tau=0$

$$\rightsquigarrow \left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = -\left[\Gamma^i_{00}\left(\frac{dx^0}{d\tau}\right)^2\right]_{\tau=0} \qquad (1.77)$$

in linearized gravity $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\Gamma^i_{00} \simeq \frac{1}{2}\left(2\partial_0 h_{0i} - \partial_i h_{00}\right)$$

but in the TT gauge $h_{0i} = h_{00} = 0 \Longrightarrow \left.\frac{d^2x^i}{d\tau^2}\right|_{\tau=0} = 0$

$\Rightarrow$ in the TT gauge $\frac{dx^i}{d\tau} = 0$ and stays zero

Particles remain at rest before/during/after the arrival of GW

The TT gauge is the frame where coordinates stretch and squeeze appropriately such that they do not change

Implementation: freely falling test masses where the masses themself are used to mark the coordinates

DETECTOR FRAME

In a lab positions are not marked by freely falling test masses. One ideally uses a "rigid ruler" to mark coordinates

$\longrightarrow$ we do <u>not</u> measure things in the TT frame!

Drop the satellite in free fall. Make measurements in that lab.

Take a sufficiently small region of size x^i

Local inertial frame:

$$ds^2 \simeq -dt^2 + \delta_{ij}dx^i dx^j$$

Expansion of $g_{\mu\nu}dx^\mu dx^\nu$ to 2nd order in x^i

$$\rightsquigarrow ds^2 \simeq -dt^2\left(1 + R_{0i0j}x^i x^j\right) - 2dt\,dx^i\left(\frac{2}{3}R_{0jik}x^j x^k\right) + dx^i dx^j\left(\delta_{ij} - \frac{1}{3}R_{ijkl}x^k x^l\right) \qquad (1.57)$$

For a detector on Earth (say LIGO...) we have an acceleration $\bar{a} = -\bar{g}$ and a rotation $\bar{\Omega}$ (Foucault's pendulum...)

$$ds^2 \simeq -dt^2 \left[1 + 2\bar{a}\bar{x} + (\bar{a}\bar{x})^2 - (\bar{\Omega}\times\bar{x}) + R_{0i0j}x^i x^j\right]$$
$$+ 2dt\,dx^i \left[\epsilon_{ijk}\Omega^j x^k - \frac{2}{3}R_{0jik}x^j x^k\right]$$
$$+ dx^i dx^j \left[\delta_{ij} - \frac{1}{3}R_{ikjl}x^k x^l\right] \qquad (1.88)$$

↳ this is called "proper lab frame" and it's where we make measurements.

$2\bar{a}\bar{x} \rightarrow$ inertial acceleration
$(\bar{a}\bar{x})^2 \rightarrow$ gravitational redshift
$(\bar{\Omega}\bar{x}) \rightarrow$ Lorentz time dilation
$\epsilon_{ijk}\Omega^j x^k \rightarrow$ Sagnac effect
$R_{ijkl} \rightarrow$ both slowly varying gravitational field and the GWs

From this metric, the geodesic equation is

$$\frac{d^2x^i}{d\tau^2} = -a^i - 2(\bar{\Omega}\times\bar{v})^i + \frac{f^i}{m} + O(x^i) \qquad (1.89)$$

gravity, Coriolis, external forces (say the LIGO suspension system)

→ centrifugal acceleration $\bar{\Omega}\times(\bar{\Omega}\times\bar{r})$ is of $O(x^i)$]
→ GWs are in the $O(x^i)$ term as well
in practice find a frequency region where the slowly varying gravitational field is not important + sufficient isolation. For LIGO, somewhere between 10Hz and 1000Hz
↝ let's focus on this frequency range,

only the GWs are present in the R tensor

Geodetic deviation powers

$$\frac{d^2\xi^i}{d\tau} + \xi^\sigma \partial_\sigma \Gamma^i_{00} \left(\frac{dx^0}{d\tau}\right)^2 = 0 \quad (1.90)$$ (at the expansion point $\Gamma^\sigma_{\mu\nu} = 0$)

$$\frac{d^2\xi^i}{d\tau^2} = -R^i_{0j0}\, \xi^j \left(\frac{dx^0}{d\tau}\right)^2$$

Mass at rest initially, then velocity $\frac{dx^i}{d\tau} = O(h)$

$$R \sim O(h)$$

$$d\tau \simeq dt\,(1+O(h))$$

so we can write to given order

$$\frac{d^2\xi^i}{dt^2} = -R^i_{0j0}\, \xi^j$$

Riemann is <u>invariant</u> (not just covariant) in linearized gravity (proof around (1.13) in MAGGIORE)

so I can compute it in the easy TT gauge

$$R^i_{0j0} = -\frac{1}{2}\ddot{h}^{TT}_{ij}$$

$$\Longrightarrow \boxed{\ddot{\xi}^i = \frac{1}{2}\ddot{h}^{TT}_{ij}\,\xi^j}$$ This is what we actually measure!

That is, <u>a Newtonian description with a force acting on the mirrors</u>

$$F_i = \frac{m}{2}\ddot{h}^{TT}_{ij}\,\xi^j$$

- Note ξ and t are coordinate separation (which we measure), not proper distance
- only in the direction where the mirror is free to move

• We have expanded everything to a "small" region where our experiment leaves. OK only if

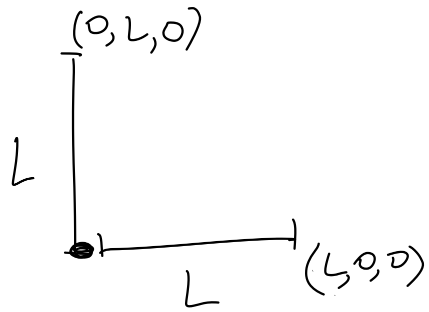
the LIGO ~ $L \ll \lambda$ → GW wavelength

OK for LIGO, not for LISA

↳ (admittedly, I don't understand the Fabry-Perot cavity business in this...)

DETECTOR RESPONSE

Compute the response of an interferometer with 90° opening angle and equal arms, like LIGO, Virgo, KAGRA



$$\ddot{\xi}^i = \frac{1}{2}\ddot{h}_{ij}\xi^j$$

→ Mirror located at $(L,0,0)$ we're interested in the motion along x only $\ddot{\xi}_x = \frac{1}{2}\ddot{h}_{xx}L$

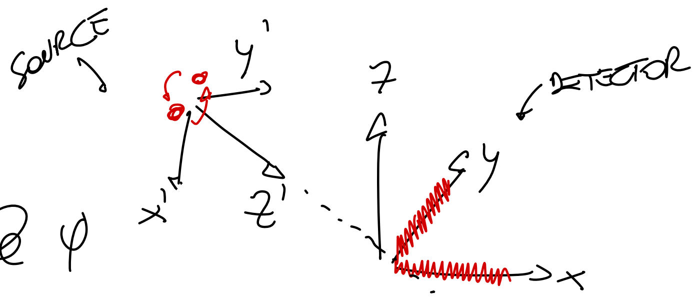
→ Mirror located at $(0,L,0)$, motion only along y $\ddot{\xi}_y = \frac{1}{2}\ddot{h}_{yy}L$

Say the GW comes from the z axis, orthogonal to the detector: we have $h_{xx} = h_+$ $h_{yy} = -h_+$ (see lecture 1)

so $h_+ = \frac{1}{2}(h_{xx} - h_{yy})$

In general, for a generic direction we must compute $\frac{1}{2}(h_{xx} - h_{yy})$...

Direction of propagation z' that forms a polar angle θ with z and an azimuthal angle φ in the plane of the detector



$$h'_{ij} = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

$$h_{ij} = R_z(\phi) R_y(\theta)\, h'_{ij}$$ rotation matrices

$$\Longrightarrow \quad h_{xx} = h_+(\cos^2\theta\cos^2\phi - \sin^2\phi) + 2h_\times \cos\theta\sin\phi\cos\phi \quad (9.133)$$

$$h_{yy} = h_+(\cos^2\theta\sin^2\phi - \cos^2\phi) - 2h_\times \cos\phi\sin\phi\cos\phi \quad (9.134)$$

$$\underbrace{h(t)} = \underbrace{\frac{1}{2}(h_{xx} - h_{yy})}_{\text{only for an interferometer}} = \frac{1}{2} h_+(1+\cos^2\theta)\cos 2\phi + h_\times \cos\theta \sin 2\phi$$

we call the response of a detector simply $h(t)$...

The frame h'_{ij} assumes that h_+ is along x'. There's an extra degree of freedom ψ, which is a rotation in the $x'y'$ plane. More rotation matrices... (see MAGGIORE problem 2.1). The result is

SEE APPENDIX 1 OF GEROSA, PRATEN, VECCHIO 2020

$$h(t) = \underbrace{F_+(\theta,\varphi,\psi)}\, h_+ + \underbrace{F_\times(\theta,\varphi,\psi)}\, h_\times$$

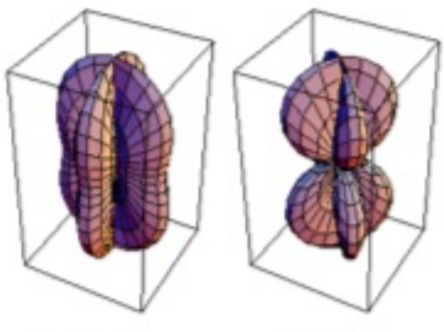
BEAM PATTERN OF A GW INTERFEROMETER

$$F_+ = \frac{1}{2}(1+\cos^2\theta)\cos 2\phi \cos 2\psi - \cos\theta \sin 2\phi \sin 2\psi$$

$$F_\times = \frac{1}{2}(1+\cos^2\theta)\cos 2\phi \sin 2\psi + \cos\theta \sin 2\phi \cos 2\psi$$

θ, φ : sky location of the source

ψ : polarization angle



+ polarization × polarization

- interferometer is maximally sensitive for a source overhead
- interferometer is blind for a source along the arm.

<u>careful</u>: do not get confused with the emission pattern of a binary...

The binary emission is

$$h_+(t) = A(t)\,\frac{1+\cos^2 i}{2}\cos\phi(t)$$
$$h_\times(t) = A(t)\,\cos i\,\sin\phi(t)$$

$A(t)$: GW amplitude
$\phi(t)$: GW phase
i : inclination

⇝ one can write (FINN CHERNOFF 1993)

$$h(t) = w\,A(t)\cos[\phi(t) - \phi_0]$$

with $w = \sqrt{\left(F_+\frac{1+\cos^2 i}{2}\right)^2 + \left(F_\times \cos i\right)^2}$ PROJECTION FACTOR

$$\phi_0 = \arctan\frac{2F_\times\cos i}{F_+(1+\cos^2 i)}$$ PHASE SHIFT (irrelevant we cannot measure it)

The various angles are all into w
One has $0<w<1$ and $w=1$ only for:
$\theta=0$: source overhead
$i=0$: source face-on