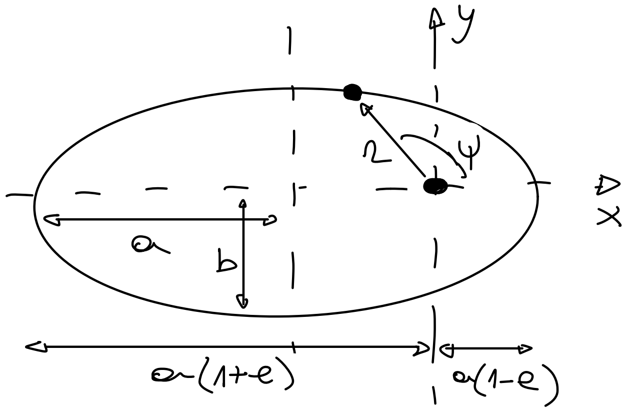
4. BINARIES (PART 2)

ECCENTRIC ORBITS

(MAGGIORE SEC 4.1.2 – 4.1.3)



Recap of Kepler:

ψ is the true anomaly

$$L = \mu r^2 \dot{\psi}$$

$$E = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\psi}) - \frac{GM}{r} =$$

$$= \frac{1}{2}\mu\dot{r}^2 + \underbrace{\frac{L^2}{2\mu r^2} - \frac{G\mu M}{r}}_{\text{effective potential}}$$

Equation of the orbit $r = \dfrac{a(1-e^2)}{1+e\cos\psi}$

$$\rightsquigarrow \dot{\psi} = \frac{\sqrt{GMa(1-e^2)}}{1+e\cos\psi}$$

orbits are periodic with $T = \dfrac{2\pi}{\omega_0}$ $\quad \omega_0^2 = \dfrac{GM}{a^3}$ Kepler's 3rd law

Cartesian coordinates centered on the focus:

$$\begin{cases} x = r\cos\psi \\ y = r\sin\psi \\ z = 0 \end{cases}$$

OK now let's compute GWs ... (4.65)

Mass quadrupole $\quad M_{ab} = \mu r^2 \begin{pmatrix} \cos^2\psi & \sin\psi\cos\psi & 0 \\ \sin\psi\cos\psi & \sin^2\psi & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ab}$

Plug this into the quadrupole formula (3.75)

$$\longrightarrow P(\psi) = \frac{8}{15}\frac{\mu^2 M^3}{a^4(1-e^2)^5}(1+e\cos\psi)^4\left[12(1+e\cos\psi)^2 + e^2\sin^2\psi\right]$$

(4.72)

GW energy is only defined when taking averages over several period of the waves. These will be a multiple of the orbital period. So let's average over T

$$P = \frac{1}{T}\int_0^T dt\, P(\psi) = \frac{1}{T}\int_0^{2\pi} \frac{d\psi}{\dot\psi} P(\psi) \quad \text{resulting integral is trivial} \tag{4.73}$$

$$\boxed{P = \frac{32}{5}\frac{\mu^2 M^3}{a^5} f(e) \qquad f(e) = \frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)}$$

POWER EMITTED ECCENTRIC ORBIT

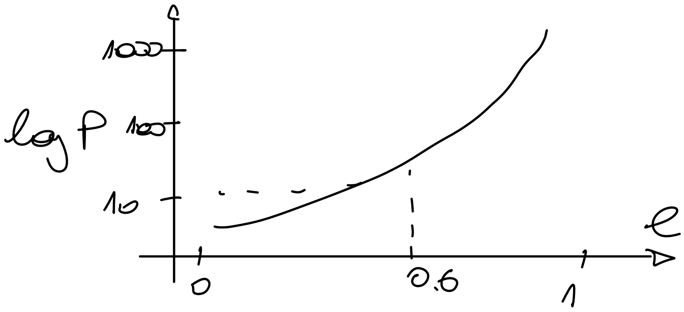
seminal result by PETERS MATHEWS 1963

(4.74_4.75)

$\rightsquigarrow$ $f(e=0)=1$ so

$$P(e) = P(e=0)\, f(e)$$

emitted power in amplified by orbital eccentricity.



period

$$\boxed{\frac{\dot T}{T} = -\frac{3}{2}\frac{\dot E}{E} = \frac{3}{2}\frac{P}{E} = -\frac{96}{5}\, 6^{5/3} \mu M^{2/3} \left(\frac{T}{2\pi}\right)^{-8/3} f(e)} \tag{4.79}$$

this is the key equation behind the HULSE_TAYLOR PULSAR, first proof of GWs! Nobel prize 1993

PSR B1913+16 $\quad m_1 \simeq 1.44\, M_\odot \quad m_2 \simeq 1.38\, M_\odot$

$e \simeq 0.617 \quad P \simeq 0.32$ days

$\rightsquigarrow v \sim 10^{-3} c$!

Discovered in 1974. Now, over a 50 yr baseline, the period decreases in spectacular agreement with the expansion above. The system is losing energy via GW emission! We are seeing the backreaction on the orbit

$\rightsquigarrow$ "INDIRECT" DETECTION OF GWs

For direct, need to wait 2015 and LIGO...

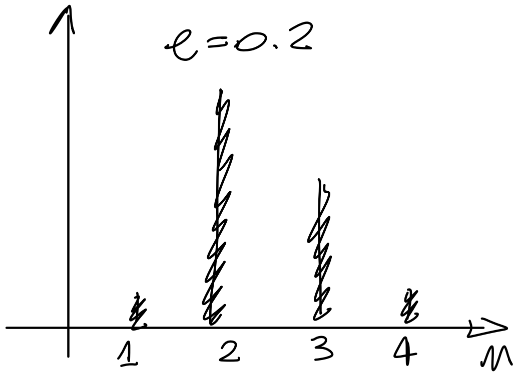
[Side note: careful that taking the parabolic limit means taking $e \to 1$ with $L \propto a(1-e^2)$ constant. Cannot send $e \to 1$ with constant a!]

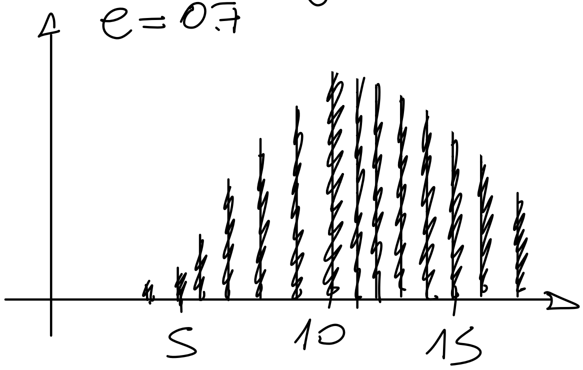
Frequency spectrum (no full calculation, sorry MAGGIORE page 181 if you want)

Emission at harmonics $\omega_n = n\,\omega_0 = \sqrt{\frac{M}{a^3}}$

$$P_n = \frac{32\,\mu^2 M^3}{5\,a^5}\, g(n,e)$$

↳ closed form using Bessel's functions

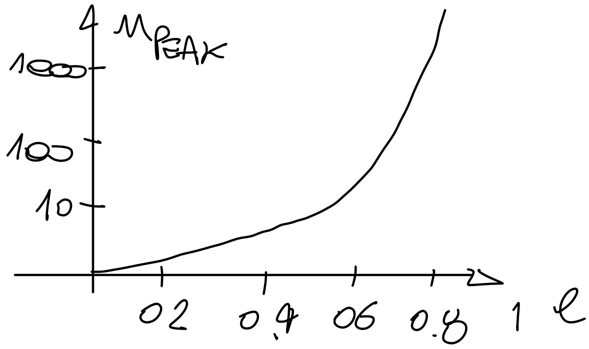




The largest contribution is not at $2\omega_0$!

WEN 2003
also
HAMERS 2021

$$n_{PEAK} \sim 2\,\frac{(1+e)^{1.2}}{(1-e^2)^{3/2}}$$



Back reaction: evolution of an eccentric orbit

We already have the energy variation (4.74-4.75)

We also need the angular momentum

$$\frac{dL^i}{dt} = \frac{2}{5}\,\epsilon^{ikl}\,\langle \ddot{Q}_{ka}\,\dddot{Q}_{la}\rangle \qquad (3.97)$$

orbital plane is in x-y, so $L = L_z$

Same calculation (remember average over one period)

$$\begin{cases} \dfrac{dE}{dt} = -\dfrac{32}{5}\,\dfrac{\mu^2 M^3}{a^5}\,\dfrac{1}{(1-e^2)^{7/2}}\left(1+\dfrac{73}{24}e^2+\dfrac{37}{96}e^4\right) \\[2ex] \dfrac{dL}{dt} = -\dfrac{32}{5}\,\dfrac{\mu^2 M^{5/2}}{a^{7/2}}\,\dfrac{1}{(1-e^2)^2}\left(1+\dfrac{7}{8}e^2\right) \end{cases}$$

Then $E = -\frac{M\mu}{2a}$ $\qquad L^2 = M\mu^2 a(1-e^2)$

$$\Longrightarrow \boxed{\begin{aligned} \frac{da}{dt} &= -\frac{64}{5}\frac{\mu M^2}{a^3}\frac{1}{(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \\ \frac{de}{dt} &= -\frac{304}{15}\frac{\mu M^2}{a^4}\frac{e}{(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right) \end{aligned}}$$

(4.116-4.117) (PETERS 1964) PETERS EQUATIONS

Some more analytical steps...

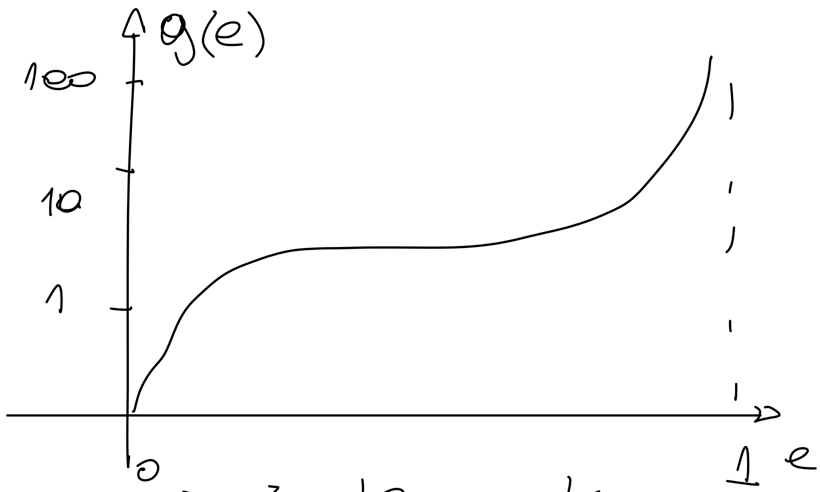
$$\frac{da}{de} = \frac{12}{19}\, a\, \frac{1+(73/24)e^2+(37/96)e^4}{e(1-e^2)\left[1+(121/304)e^2\right]}$$

$$\Longrightarrow a(e) = c_0 \frac{e^{12/19}}{1-e^2}\left(1+\frac{121}{304}e^2\right)^{870/2299}$$

determined by the initial conditions

(This formulation is mostly never implemented numerically because it's discontinuous in the $e\to 0$ limit. We presented a regularized formulation in FUMAGALLI GEROSA 2023)

$$a(e) = a_0 \frac{g(e)}{g(e_0)}$$



Note that this equation predicts that $e \to 1$ for $a \to \infty$ so all orbits were parabolic in the past. This is not true! We averaged over an orbit, and you can't do it for $e\to 1$ where the period diverges $\rightsquigarrow$ NON ADIABATIC EFFECTS...

Crucially we has $e \to 0$ for $a \to 0$
that is BINARIES CIRCULARIZE AS THEY INSPIRAL

We expect (and observe!) most GW sources to be close to circular

For $e \ll 1$ $\rightsquigarrow$ $a(e) \simeq \frac{a_0}{g(e_0)} e^{12/19}$

this exponent is <1. That means that the eccentricity decreases faster than the orbital separation. If there's an inspiral, binaries must circularize

For $e \sim 1$ $\rightsquigarrow$ $a(e) \simeq \frac{a_0}{g(e_0)} \frac{1}{1-e^2}$ ... quickly!

Time to merger

For the circular case we find that

$$\tau_0 = \int_{a_0}^{0} \left(\frac{da'}{dt}\right)^{-1} da' = \frac{5}{256} \frac{a_0^4}{M^2 \mu} \qquad (4.132)$$

Now we have (at merger $e \to 0$ ...)

$$\tau_0 = \int_{e_0}^{0} \left(\frac{de'}{dt}\right)^{-1} de' = \qquad \text{(need to use } a(e)\text{)}$$

$$= \underbrace{\frac{5}{256} \frac{a_0^4}{M^2 \mu}}_{\tau_c^{(\mathrm{CIRCULAR})}} \underbrace{\frac{48}{19} \frac{1}{g^4(e_0)} \int_0^{e_0} \frac{g^4(e') (1-e'^2)^{5/2}}{e' (1 + \frac{121}{304} e'^2)} de'}_{F(e_0)}$$

In practice $F(e_0) = (1-e_0)^{7/2} \boxed{}$ factor between 1 and 1.8

Eccentric binaries merge faster

COSMOLOGY AND GW BINARIES

Recap of FLRW geometry

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi\right) \qquad (4.141)$$

$a(t)$: scale factor

$K=0$: flat universe only

(t, r, θ, ϕ) : comoving coordinates

$d\eta = \frac{dt}{a(t)}$ conformal time normalized such that $\eta = t$ today

definition redshift $\quad 1+z = \frac{a(t_{OBS})}{a(t_{EMIS})}$

$$\begin{cases} dt_{DET} = (1+z)\,dt_S \\ f_{DET} = \frac{f_S}{1+z} \\ \lambda_{DET} = (1+z)\,\lambda_S \end{cases}$$

S = "source frame"

DET = "detector frame"

$d_L = (1+z)\, a(t_0)\, r$ LUMINOSITY DISTANCE

t_0 ↳ present time

Scalar field propagation

Let's start from the simpler case of a scalar quantity ϕ, propagating in FLRW (hint: GWs then are basically the same...)

$$\Box \phi = 0 \qquad \Box = D_\mu D^\mu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\, g^{\mu\nu}\partial_\nu\right)$$

Ansatz $\quad \phi = \frac{1}{a(t)\, r}\, g(t, r) \quad$ then use η

$$\rightsquigarrow \frac{d^2 g}{dr^2} - \frac{d^2 g}{d\eta^2} + \frac{g}{a}\frac{d^2 a}{d\eta^2} = 0 \qquad (4.180)$$

Let's assume $\omega^2 \gg \frac{1}{\eta^2}$. Very reasonable, it means $2\pi f_{GW} \gg \frac{1}{t_{HUBBLE}}$ i.e. GWs with wavelength smaller than the size of the observable universe.

$$\frac{g}{a}\frac{d^2a}{d\eta^2} \sim \frac{g}{\eta^2} \quad \text{negligible compared to} \quad -\frac{d^2g}{d\eta^2} = \omega^2 g$$

solution is $\quad g(r,\eta) \simeq e^{\pm i\omega(\eta - r)} \quad (4.181)$

$$\phi(r,\eta) \simeq \frac{1}{r\,a(\eta)}\, g(\eta - r) \qquad \eta(t_0) = t \quad \text{today}$$

$$\phi(r,t) \simeq \frac{1}{r\,a(t)}\, g(t-r) \qquad (4.183)$$

compared to the usual solution in flat spacetime, we need to replace $r \to r\,a(t)$

GW propagation

This is what we had (indicating "source frame")

$$h_+(t_s) = h_c(t_s^{RET})\,\frac{1+\cos^2 i}{2}\cos\left(2\pi\int f_{GW,s}(t_s')\,dt_s'\right) \quad (4.170)$$

$$h_\times(t_s) = h_c(t_s^{RET})\,\cos i\,\sin\left(2\pi\int f_{GW,s}(t_s')\,dt_s'\right) \quad (4.171)$$

$$h_c = \frac{4}{r} M_c^{5/3}\left[\pi f_{GW,s}(t_s^{RET})\right]^{2/3} \quad \begin{matrix}(4.172)\\(4.184)\end{matrix}$$

Full calculation using $g_{\mu\nu} = (FLRW)_{\mu\nu} + h_{\mu\nu}$ ...

... the polarizations do not mix. so it's like two scalar fields as above. All I have to do is replace $r \to a(t_{DET})\,r$ in (4.172). That's all...

$$h_c = \frac{4}{a(t_{DET})\,r} M_c^{5/3}\left[\pi f_{GW,s}(t_s^{RET})\right]^{2/3}$$

Rewrite more conveniently using "detector frame" quantities

$$\int f_{GW,s}\,dt_s = \int f_{GW,DET}(1+z)\,\frac{dt_{DET}}{(1+z)} = \int f_{GW,DET}\,dt_{DET} \quad (4.185)$$

$$\rightsquigarrow h_c = \frac{4}{d_L(z)} (1+z)^{5/3} M_c^{5/3} (\pi f_{GW,DET})^{2/3} = \frac{4}{d_L} \left[(1+z) M_c\right]^{5/3} (\pi f_{GW,DET})^{2/3}$$

$(1+z)^{2/3}$ coming from f_{GW}
$(1+z)$ coming from d_L

And the frequency evolution is

$$f_{GW,DET} = \frac{1}{1+z} f_{GW,S} \underset{(4.19)}{=} \frac{1}{1+z} \frac{1}{\pi} \left(\frac{5}{256} \frac{1+z}{\tau_{DET}} \right)^{3/8} M_c^{-5/8}$$

$$= \frac{1}{\pi} \left(\frac{5}{256} \frac{1}{\tau_{DET}} \right)^{3/8} \left[(1+z) M_c \right]^{-5/8}$$

That is, everything is the same but

$$\boxed{\begin{array}{l} \text{DISTANCE } r \longrightarrow \text{LUMINOSITY DISTANCE } d_L \\ \text{CHIRP MASS } M_c \longrightarrow \text{"REDSHITED CHIRP MASS } (1+z) M_c \end{array}}$$

This is all one needs to remember...

In GW astronomy, we measure d_L and $(1+z) M_c$
we do not have access to the source properties.

In practice:

- measure d_L, assume a cosmology to determine z and use that value of z to measure M_c.
- With a counterpart: measure d_L from GW, measure z from light $\longrightarrow$ constrain the cosmology $d_L(z)$
  "STANDARD SIRENS" (analogy with standard candles)

Note: the scale of gravity is $\frac{GM}{c^3}$. All we are doing is redshifting that scale to $(1+z) \frac{GM}{c^3}$, which is a very natural result.