3. BINARIES (PART 1)

We now apply this generic formalism to the case of binary systems

FIXED, CIRCULAR ORBIT

(MAGGIORE PROBLEM 3.2)

Start easy. Two masses on a fixed circular orbit

neglect backreaction for now

In the center of mass frame, equivalent to the motion of μ around M.

total mass $M = m_1 + m_2$ reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$

$$x_0(t) = R\cos(\omega_s t + \pi/2)$$
$$y_0(t) = R\sin(\omega_s t + \pi/2) \qquad (3.324)$$
$$z_0(t) = 0$$

arbitrary, just doing as the book...

frequency of the SOURCE

the mass quadrupole is $M^{ij} = \mu x_0^i(t) x_0^j(t)$

This is because for two particles $x_1(t), x_2(t)$

$$M^{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = m X_{CM}^i X_{CM}^j + \mu x_0^i x_0^j = \mu x_0^i x_0^j$$

center of mass $\bar{X}_{CM} = \frac{m_1 \bar{x}_1 + m_2 \bar{x}_2}{m_1 + m_2}$ choose frame $\bar{X}_{CM} = 0$

Components are:

$$M_{11} = \mu R^2 \frac{1 - \cos(2\omega_s t)}{2} \qquad M_{22} = \mu R^2 \frac{1 + \cos(2\omega_s t)}{2} \qquad (3.325\_327)$$

$$M_{12} = -\frac{1}{2}\mu R^2 \sin(2\omega_s t) \qquad M_{31} = M_{32} = M_{33} = 0$$

$\rightsquigarrow$ from here, one immediately see that the GW frequency must be $\boxed{\omega_{GW} = 2\omega_s}$

Plug into (3.72): (3.330_331)

$$h_+(t,\theta,\phi) = \frac{1}{r}\, 4\mu\omega_s^2 R^2 \left(\frac{1+\cos^2\theta}{2}\right) \cos(2\omega_s t_{RET} + 2\phi)$$

$$h_\times(t,\theta,\phi) = \frac{1}{r}\, 4\mu\omega_s^2 R^2 \cos\theta \sin(2\omega_s t_{RET} + 2\phi)$$

- Angular dependence on the binary inclination
  - $\theta = 0$ face-on: circular polarization $h_+, h_\times$
  - $\theta = \frac{\pi}{2}$ edge-on: linear polarization h_+
- Note only the combination $(\omega_s t_{RET} + \phi)$ is important, reflecting the symmetry of the source. Azimuthal angle degenerate with time, set $\phi = 0$

Last lecture, POWER EMITTED in the quadrupolar approximation

(3.73_3.336)

$$\frac{dP}{d\Omega} = \frac{r^2}{16\pi} \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle$$

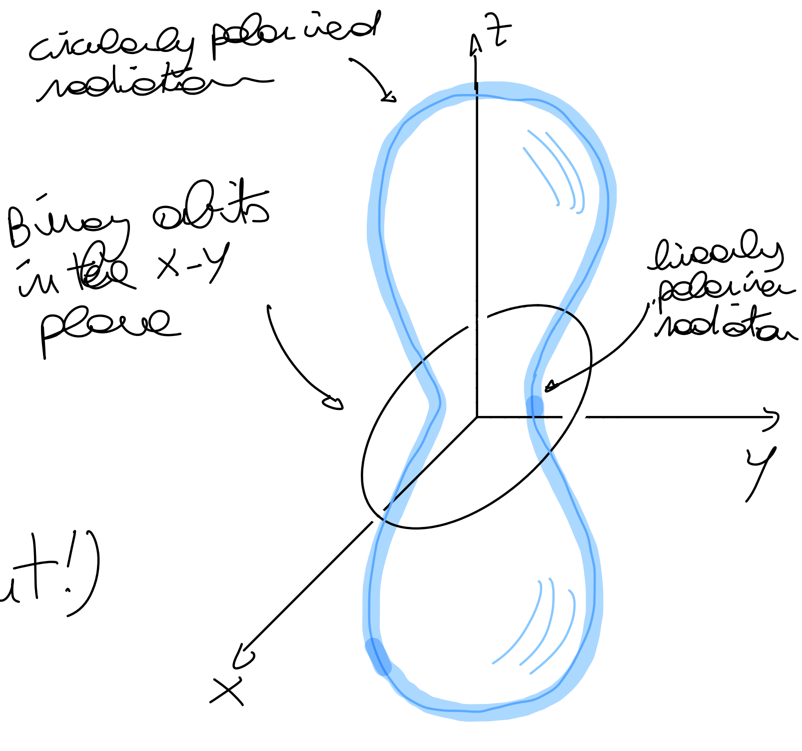
Average over one orbit $\langle \cos^2(2\omega_s t)\rangle = \langle \sin^2(2\omega_s t)\rangle = \frac{1}{2}$

$$\rightarrow \boxed{\frac{dP}{d\Omega} = \frac{2\mu^2 R^4 \omega_s^6}{\pi}\left[\left(\frac{1+\cos^2\theta}{2}\right)^2 + \cos^2\theta\right]}$$

GW EMISSION PATTERN

Emission is largest in the direction $\perp$ to the orbital plane i.e. along the orbital angular momentum

(for BH with spins, this direction is not constant!)



TOTAL EMITTED POWER

$$P = \frac{32}{5}\mu^2 R^4 \omega_s^6 = \frac{32}{5}\mu^2\frac{v^6}{R^2} = \frac{32}{5}\mu^2\frac{M^6}{R^5}$$

$v = \omega_s R$ $\qquad$ $\omega_s^2 = \frac{M}{R^3}$ Kepler

We need binaries that are massive, close, and fast...

ENERGY RADIATED in one period $T = \frac{2\pi}{\omega_s}$

$$E = \frac{64\pi}{5}\frac{G\mu^2}{R}\left(\frac{v}{c}\right)^5 \qquad (3.340)$$

the energy scale of the problem is that of gravity $\frac{G\mu^2}{R}$

Then GWs are very small, suppressed by a factor $\left(\frac{v}{c}\right)^5$, i.e. they enter the dynamics at 2.5PN order

CHIRP MASS

How do the masses enter the strain (3.330-331)?
Remove R^2 with Kepler $\omega^2 = M/R^3$

then define

$$\boxed{M_c = \mu^{3/5} M^{2/5} = \frac{(m_1 m_2)^{3/5}}{(m_1+m_2)^{1/5}}} \quad \text{chirp mass} \qquad (4.2)$$

$$f_{GW} = \frac{\omega_{GW}}{2\pi} = \frac{\omega_s}{\pi} \qquad (4.3)$$

$$h_+(t) = \frac{4}{r} M_c^{5/3} (\pi f_{GW})^{2/3} \frac{1+\cos^2\theta}{2} \cos(2\pi f_{GW} t_{RET})$$

$$\Longrightarrow \quad h_\times(t) = \frac{4}{r} M_c^{5/3} (\pi f_{GW})^{2/3} \cos\theta \sin(2\pi f_{GW} t_{RET})$$

and $P = \frac{32}{5}\left(M_c \pi f_{GW}\right)^{2/3}$

The masses only entered together with M_c at leading order. This is why the chirp mass is the best measured mass combination in GW astronomy

HIGHER ORDER

This is only quadrupole radiation and happens at $\omega_{GW} = 2\omega_s$

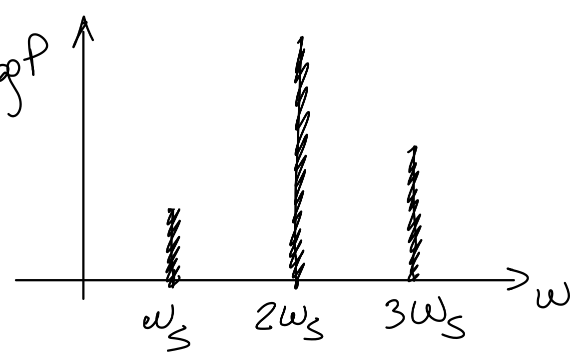
The next-to-leading order are the mass octupole and the current quadrupole. Those modes are emitted at $\omega_{GW} = \omega_s$ and $\omega_{GW} = 3\omega_s$ $(\delta m = m_1 - m_2)$

$$P(\omega_s) = \frac{25}{896}\left(\frac{v}{c}\right)^2\left(\frac{\delta m}{m}\right)^2 P_{QUAD}(2\omega_s) \qquad (3.350)$$

$$P(3\omega_s) = \frac{1215}{896}\left(\frac{v}{c}\right)^2\left(\frac{\delta m}{m}\right)^2 P_{QUAD}(2\omega_s) \qquad (3.351)$$

suppressed by one PN order

Suppressed for equal mass binaries $\delta m = 0$
(see e.g. event GW190412. crucial for EMRIs in LISA)



in practice, a spectrum of multiple harmonics (even for a circular binary on fixed orbit)

The mass ratio $q = \frac{m_2}{m_1}$ (or equivalently δm) and not just M_c enters the waveform

CIRCULAR ORBIT, WITH BACKREACTION

This was for fixed orbits, which is ok if

$$t_{INSPIRAL} \sim E\left(\frac{dE}{dt}\right)^{-1} \gg t_{OBSERVATION}$$

This is the case for some white dwarf binaries in LISA, but definitely not for BH binaries in LIGO. Need to include the evolution of the orbit.

$$E_{ORBIT} = -\frac{G m_1 m_2}{2R} = -\left(\frac{M_c^5 \omega_{GW}^2}{32}\right)^{1/3} \quad (4.16)$$

GW dissipate energy, E_{ORBIT} becomes more negative, R decreases. But $P \propto \frac{1}{R^2}$ so if R decreases, GWs carry away even more energy

→ THE TWO-BODY PROBLEM IN GR IS UNSTABLE

Radial velocity (4.15)

$$\text{Kepler} \quad R^3 = \frac{M}{\omega_s^2} \longrightarrow \underbrace{\dot{R}}_{\text{radial velocity}} = -\frac{2}{3} \underbrace{(\omega_s R)}_{\text{tangential velocity } v = \omega_s R} \frac{\dot{\omega}_s}{\omega_s^2}$$

if $\dot{\omega}_s \ll \omega_s^2 \Longrightarrow |\dot{R}| \ll v$

QUASI CIRCULAR APPROXIMATION

↳ Model as a series of circular orbit in a quasi-adiabatic fashion

$$-\frac{dE_{ORBIT}}{dt} = P \Longrightarrow \dot{f}_{GW} = \frac{96}{5}\pi^{8/3} M_c^{5/3} f_{GW}^{11/3} \quad (4.18)$$

frequency diverges in finite time

define t_{COAL} : $f_{GW}(t_{COAL}) = \infty$

$$\tau = (t_{COAL})_{RET} - t_{RET} = t_{COAL} - t$$

$$\Longrightarrow f_{GW}(\tau) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau}\right)^{3/8} M_c \quad (4.19)$$

[Some numbers...

$M_c = 10\, M_\odot$

$\tau = 1\,s$ before merger

$\longrightarrow f_{GW}(\tau) \sim 100\, Hz$

$r = 10\, Mpc$ (distance to the Virgo cluster)

$\longrightarrow h_{+,\times} \sim 10^{-21}$ (from (4.3))

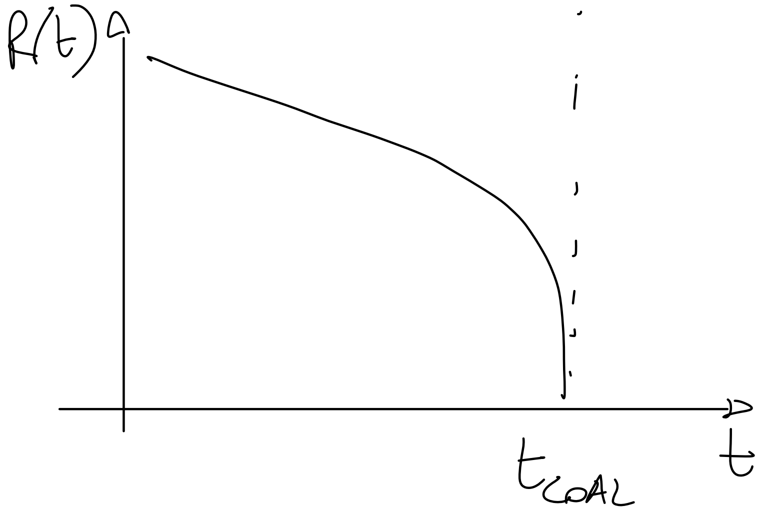
LIGO targets $\sim 100 Hz$ with a strain sensitivity of 10^{-21}!]

Separation evolves as

$$\frac{\dot{R}}{R} = -\frac{2}{3}\frac{\dot{\omega}_{GW}}{\omega_{GW}} = -\frac{1}{4\tau} \Longrightarrow R(\tau) = R_0\left(\frac{\tau}{\tau_0}\right)^{1/2}$$

where $\boxed{\tau_0 = \frac{5}{256}\frac{R_0^4}{M^2\mu}}$ INSPIRAL TIMESCALE

plug Kepler's law into (4.19)



At $t = t_{COAL}$ ($\tau = 0$) all our approximations break down, we can't really go there analytically

$\tau_0 \propto R_0^4$ very steep power!

→ if R_0 is small, τ_0 is very small then when we observe BH binaries, their dynamics is entirely driven by GW emission. The astrophysical context is irrelevant! Vacuum is a great approximation

→ if R_0 is large, τ_0 is very large

- for stellar mass BHs $M \sim 10 M_\odot$

$$\tau_0 \gtrsim 10^{10}\,\text{yr} \quad \text{if } R_0 \gtrsim 1 R_\odot$$

Hubble time

this is much less than the typical separation of binary stars

→ FORMATION CHANNEL PROBLEM OF GW ASTRONOMY

- For supermassive BHs $M \sim 10^6 M_\odot$

$$\tau \gtrsim 10^{10}\,\text{yr} \quad \text{if} \quad R_0 \gtrsim 0.01\,\text{pc}$$

but galactic dynamics can only bring BHs down to $\sim 1\,\text{pc}$

$\longrightarrow$ "FINAL PARSEC PROBLEM" (arguably never solved)

Another consequence of $R(t) \propto \sqrt{\tau}$ is that the binary spends more time at large separation, which corresponds to lower frequenc

WAVEFORM How does the emission change? Qualitatively: R goes down, frequency goes up

this is called a "CHIRP"

Promote $R \to R(t)$
$\omega_s \to \omega_s(t)$

GW phase

$$\phi(t) = 2\int_{t_0}^{t} dt'\, \omega_s(t') = \int dt'\, \omega_{GW}(t') \qquad (4.28)$$

Same calculation as before but
- Replace $\omega_{GW} \times t \to \phi(t)$ in the phase
- Replace $\omega_{GW} \longrightarrow \omega_{GW}(t)$ in the amplitude

There should be contributions from $\dot{R}$ and $\dot{\omega}_{GW}$ which we can neglect in our quasi-adiabatic approximation

$$h_+(t) = \frac{4}{r} M_c^{5/3} \left(\pi f_{GW}(t_{RET})\right)^{2/3} \left(\frac{1+\cos^2\theta}{2}\right) \cos\phi(t_{RET}) \qquad (4.29)$$
$$h_\times(t) = \frac{4}{r} M_c^{5/3} \left(\pi f_{GW}(t_{RET})\right)^{2/3} \cos\theta \sin\phi(t_{RET})$$

Integrate (4.19) with $d\tau = -dt$

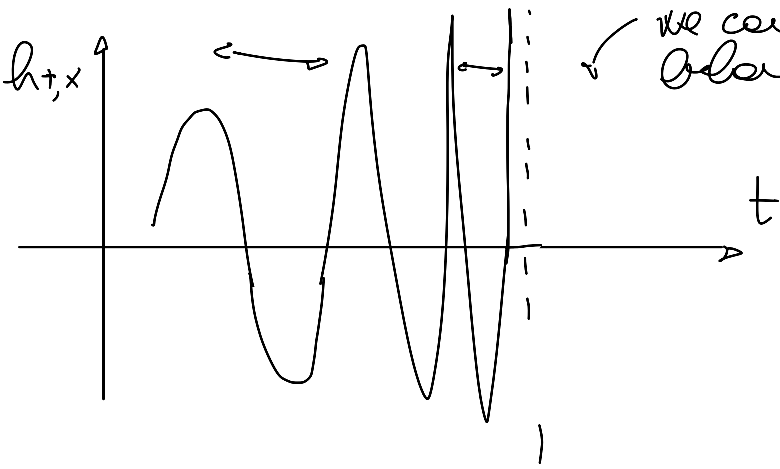
$$\phi(\tau) = -2(5M_c)^{-5/8} \tau^{5/8} + \phi_0 \qquad (4.30)$$

$\hookrightarrow \phi(\tau=0)$ phase at coalescence

some trigonometry...

$$h_+(t) = \frac{1}{r} M_c^{5/4} \left(\frac{5}{\tau}\right)^{1/4} \left(\frac{1+\cos^2\theta}{2}\right) \cos\phi(\tau) \qquad (4.31\_32)$$
$$h_\times(t) = \frac{1}{r} M_c^{5/4} \left(\frac{5}{\tau}\right)^{1/4} \cos i \sin\phi(\tau)$$

That is, both the amplitude (from 4.31_32) and the frequency (from 4.19) go up as the binary approaches merger ($\tau \to 0$)



we can't capture the late time behavior with this formalism

this is the typical "GW CHIRP SIGNAL" you've probably seen a million times...

→ Remember:
this is only the leading_order (quadrupolar) radiation
we are assuming circular orbits
we are neglecting BH spins
we are neglecting non_adiabatic effects

This chain of approximations can be fully formalized with a full POST_NEWTONIAN EXPANSION