2. GW EMISSION

QUADRUPOLE EMISSION

MAGGIORE CHAP 3

We need a non-relativistic limit $v \ll c$
For a two-body system with total mass M
and reduced mass μ

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\mu \frac{GM}{r} \longrightarrow \left(\frac{v}{c}\right)^2 = \frac{GM/c^2}{r} = \frac{M}{r}$$

$\rightsquigarrow$ $v \ll c$ corresponds to $r \gg M$ large orbital separation

Back to linearized gravity

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \qquad (1.24)\ (3.3) \qquad x = (t, \bar{x})$$

Solve with Green's function

$$\Box G(x-x') = \delta^4(x-x') \qquad (3.4)$$

$$G(x-x') = -\frac{1}{4\pi}\frac{1}{|\bar{x}-\bar{x}'|}\,\delta\Big(\underbrace{t - \frac{|\bar{x}-\bar{x}'|}{c}}_{\text{retarded time}} - t'\Big)$$

$$\Longrightarrow \bar{h}_{\mu\nu} = -16\pi \int d^4x'\, G(x-x')\, T_{\mu\nu}(x')$$

(3.8)
$$= 4\int d^3\bar{x}' \frac{1}{|\bar{x}-\bar{x}'|} T_{\mu\nu}(t - |\bar{x}-\bar{x}'|, \bar{x}')$$

wave at time t depends on the property of the source at time $t - |\bar{x}-\bar{x}'|$

go to the TT gauge now

$$h^{TT}_{ij} = \Lambda_{ijkl} h_{kl} = \Lambda_{ijkl} \bar{h}_{kl}$$

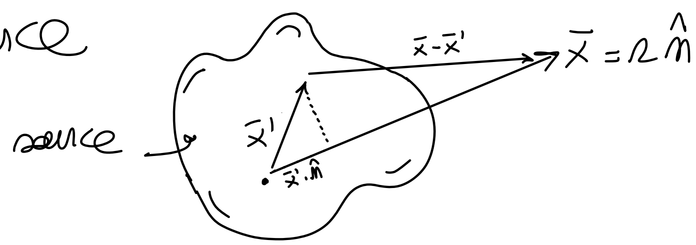
property of Λ : $\Lambda_{iikl} = \Lambda_{ijkk} = 0$

$$h^{TT}_{ij}(t,\bar{x}) = 4\Lambda_{ijkl}(\hat{x}) \int d^3\bar{x}' \frac{1}{|\bar{x}-\bar{x}'|} T_{kl}(t - |\bar{x}-\bar{x}'|, \bar{x}') \qquad (3.9)$$

Outside the source

$\hat{x} = \hat{n} \quad |\bar{x}| = r$

$|\bar{x}-\bar{x}'| \sim r - \bar{x}'\cdot\hat{n}$



$$h^{TT}_{ij} = \frac{4}{r} \Lambda_{ij,kl}(\hat{n}) \int d^3x' \, T_{kl}(t - r + \bar{x}'\hat{n}, \bar{x}') \quad (3.11)$$

integral over the source

To understand the non-relativistic limit it's better to look at this in Fourier space

$$T_{kl}(t - r + \bar{x}'\hat{n}, x') = \int \frac{d^4k}{(2\pi)^4} \tilde{T}_{kl}(\omega, \bar{k}) \, e^{-i\omega(t - r + \bar{x}'\hat{n}) + i\bar{k}\bar{x}'}$$

ω is the frequency of the radiation

$\omega \sim \frac{v}{d}$ $\rightarrow$ velocity of the source
$\phantom{\omega \sim \frac{v}{}}$ $\rightarrow$ size of the source

$\bar{x}' \sim d$ because it covers the source

so $\omega \bar{x}'\hat{n} \sim v$. Taking $v \ll 1$ means expanding

$$e^{-i\omega(t - r + \bar{x}'\hat{n})} = e^{-i\omega(t - r)}\left(1 - i\omega x'^i n^i + \dots\right) \quad (3.29)$$

In the frequency domain a multiplication by ω is a derivative in the time domain

$$T_{kl}(t - r + \bar{x}'\hat{n}, \bar{x}') \simeq T_{kl}(t - r, \bar{x}') + x'^i n^i \partial_t T_{kl} + \dots \quad (3.30)$$

derivative evaluated at $(t - r, x')$

Now plug (3.30) into (3.11)

$$h^{TT}_{ij} = \frac{1}{r} 4 \Lambda_{ij,kl}(\hat{n}) \left[S^{kl} + \underbrace{n_m \dot{S}^{klm}}_{O\left(\frac{d}{t}\right) \sim O\left(\frac{v}{c}\right)} + \underbrace{\frac{1}{2} n_m n_p \ddot{S}^{klmp}}_{O\left(\frac{v^2}{c^2}\right)} \dots \right]_{\text{ret}} \quad (3.34)$$

evaluated at $t - r/c$

this is a Post-Newtonian expansion
corrections going as $\left(\frac{v}{c}\right)^n$

Here $S^{ij}(t) = \int d^3\bar{x}\, T^{ij}(t,\bar{x})$
$S^{ijk}(t) = \int d^3\bar{x}\, T^{ij}(t,\bar{x})\, x^k$ } moments of the stress energy tensor

Now rewrite

$$M = \int d^3x\, T^{00} \qquad M^i = \int d^3x\, T^{00} x^i \qquad M^{ij} = \int d^3x\, T^{00} x^i x^j$$

$$P^i = \int d^3x\, T^{0i} \qquad P^{ij} = \int d^3x\, T^{0i} x^j \quad \text{etc...} \qquad (3.35\text{-}3.41)$$

For instance

$$\dot{M} = \frac{\partial M}{\partial t} = \int_V d^3x\, \partial_0 T^{00} = -\int_V d^3x\, \partial_i T^{0i} = -\int_{\partial V} dA^i\, T^{0i} = 0$$

$\partial_\mu T^{\mu 0} = 0$

volume larger than the source $T^{\mu\nu} = 0$ on ∂V

Similarly:

$\dot{M} = 0$ mass conservation (3.45-3.51)

$\dot{M}^i = P^i$ momentum equation

$\dot{M}^{ij} = P^{ij} + P^{ji}$ mass quadrupole

$\dot{P}^i = 0$ linear momentum conservation

$\dot{P}^{ij} = S^{ij} \Longrightarrow \dot{P}^{ij} - \dot{P}^{ji} = S^{ij} - S^{ji} = 0$
angular momentum conservation

The strategy now is replacing S^{kl} in (3.34) with M's and P's

$$S^{ij} = \frac{1}{2}\ddot{M}^{ij} \rightarrow \text{mass quadrupole} \qquad (3.52)$$

$$\dot{S}^{klm} = \frac{1}{6}\dddot{M}^{ijk} + \frac{1}{3}\left(\ddot{P}^{ijk} + \ddot{P}^{jik} - 2\ddot{P}^{kij}\right) \qquad (3.53)$$

mass octupole — current quadrupole

We only look at the dominant term (Sec 3.3)

$$\boxed{h^{TT}_{ij}(t,\vec{x}) = \frac{2}{r}\Lambda_{ijkl}(\hat{n})\ddot{M}^{kl}(t-r)}$$

GWs are the second derivative of the mass quadrupole

$\rho = T^{00}$ mass density

define $Q^{ij} = M^{ij} - \frac{1}{3}\delta^{ij}M_{kk} = \int d^3x\, \rho(t,\vec{x})(x^i x^j - \frac{1}{3}r^2\delta^{ij})$

↳ Traceless mass quadrupole moment

$$h^{TT}_{ij} = \frac{2}{r}\Lambda_{ijkl}\left(\ddot{Q}^{kl} - \frac{1}{3}\delta^{kl}\ddot{M}_{\alpha\alpha}\right) = \frac{2}{r}\Lambda_{ijkl}\ddot{Q}^{kl}(t-r)$$

$$= \frac{2}{r}\ddot{Q}^{TT}_{ij}(t-r) \qquad \Lambda_{ijkl}\delta^{kl} = 0 \qquad (3.59)$$

But we have Q not Q^{TT}... to get the GW in an arbitrary direction we could compute $\Lambda_{ijkl}\ddot{Q}^{kl}$ which is a pain. Something easier:

$$\hat{n} = \hat{z} \rightarrow P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \Lambda_{ijkl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$$

for a generic A_{kl} one has

$$\Lambda_{ijkl}A_{kl} = (PAP)_{ij} - \frac{1}{2}P_{ij}\,\mathrm{tr}(PA)$$

$$PAP = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \mathrm{Tr}(PA) = A_{11} + A_{22}$$

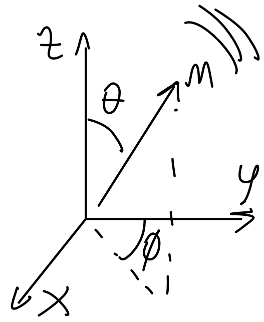
$$\Longrightarrow \Lambda_{ijkl}\ddot{M}_{kl} = \begin{pmatrix} \frac{\ddot{M}_{11} - \ddot{M}_{22}}{2} & \ddot{M}_{12} & 0 \\ \ddot{M}_{12} & -\frac{(\ddot{M}_{11} - \ddot{M}_{22})}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}$$

Read off the polarizations for GW propagation along z

$$h_+ = \frac{1}{r}(\ddot{M}_{11} - \ddot{M}_{22}) \qquad (3.65)$$

$$h_\times = \frac{2}{r}\ddot{M}_{12} \qquad (3.66)$$

For the generic case now apply a rotation



$$n = (\sin\theta \sin\phi, \sin\theta\cos\phi, \cos\theta)$$

$$n' = (0,0,1)$$

$$n_i = R_{iJ} n'_J \qquad R = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

$$M_{iJ} = R_{ik} R_{Jl} M'_{kl}$$

the result is (3.72)

$$\boxed{\begin{aligned} h_+(t,\theta,\phi) = \frac{1}{2}\Big[& \ddot{M}_{11}(\cos^2\phi - \sin^2\phi\cos^2\theta) + \ddot{M}_{22}(\sin^2\phi - \cos^2\phi\cos^2\theta) \\ & - \ddot{M}_{33}\sin^2\theta - \ddot{M}_{12}\sin 2\phi\,(1+\cos^2\theta) \\ & + \ddot{M}_{13}\sin\phi\sin 2\theta + \ddot{M}_{23}\cos\phi\sin 2\theta \Big] \\ h_\times(t,\theta,\phi) = \frac{1}{2}\Big[& (\ddot{M}_{11} - \ddot{M}_{22})\sin 2\phi\cos\theta + 2\ddot{M}_{12}\cos 2\phi\cos\theta \\ & - 2\ddot{M}_{13}\cos\phi\sin\theta + 2\ddot{M}_{23}\sin\phi\sin\theta \Big] \end{aligned}}$$

These allow to compute GWs in the leading-order approximation given the mass quadrupole ~~of~~ of the source. They encode the EMISSION OF GWs

<u>COMMENT</u> Why not monopole and/or dipole radiation?

Mass and momentum are conserved $\dot{M} = \dot{P}^i = 0$ and h^{TT}_{iJ} depends on derivatives...

This is cheating! $\dot{M} = \dot{P}^i = 0$ only in the linearized theory. Actually, the system is emitting GWs, so the <u>mass is changing</u>.

The absence of monopolar and dipolar

radiation calcs in PN theory, where one builds moments of a more generic quantity involving $t_{\mu\nu}$ (GW energy...)

RADIATED ENERGY from (1.153)

$$\frac{dP}{d\Omega} = \frac{r^2}{32\pi} \langle \dot{h}^{TT}_{ij} \dot{h}^{TT}_{ij} \rangle = \frac{1}{8\pi} \Lambda_{ijkl}(\hat{n}) \langle \dddot{Q}_{ij} \dddot{Q}_{kl} \rangle$$

average over several GW periods

derivatives evaluated at the retarded time

$$\int d\Omega \, \Lambda_{ijkl}(\hat{n}) = \frac{2\pi}{15} \left(11 \delta_{ik}\delta_{jl} - 4\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} \right) \quad (3.74)$$

$$\boxed{P = \frac{G}{5c^5} \langle \dddot{Q}_{ij} \dddot{Q}_{ij} \rangle}$$

QUADRUPOLE FORMULA (3.75)

$$= \frac{G}{5c^5} \langle \dddot{M}_{ij} \dddot{M}_{ij} - \frac{1}{3} \dddot{M}_{kk}^2 \rangle$$

Power emitted in GWs

Similarly one can show that the radiated angular momentum is

$$\frac{dJ^i}{dt} = \frac{2}{5} \epsilon^{ikl} \langle \ddot{Q}_{ka} \dddot{Q}_{la} \rangle \quad (3.97)$$

there's also a linear momentum emitted, which causes the so called BLACK HOLE KICKS

One needs a full multipolar expansion, see THORNE (1980), RUIZ et al (2008), GEROSA et al (2018)

modern implementation of these fluxes

RADIATION REACTION

What happens to the source while GWs are being emitted? (SEC 3.3.4)

Equate GW flux at t to source behaviour at the retarded time

$$\left.\frac{dE_{SOURCE}}{dt}\right|_{RET\ TIME} = -\left.\frac{dE_{GW}}{dt}\right|_{TIME} = -\frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle_{RET}$$

$$\rightsquigarrow \frac{dE_{SOURCE}}{dt} = -\frac{1}{5}\langle \dddot{Q}_{ij}\dddot{Q}_{ij}\rangle \text{ at generic } t$$

Newtonian formalism

$$\frac{dE_{SOURCE}}{dt} = \langle F_i v_i\rangle = \langle \int d^3x' \frac{dF_i}{dV}\dot{x}'_i$$

"force" "force" density

inside $\langle\cdot\rangle$ we can integrate by parts and neglect boundary terms (remember we are averaging out fast oscillations)

$$\frac{dE_{SOURCE}}{dt} = -\frac{1}{5}\langle \dot{Q}_{ij}\frac{d^5 Q_{ij}}{dt}\rangle \qquad (3.103)$$

$$\frac{dQ_{ij}}{dt} = \int d^3x' \, \partial_t T^{00}(t,x')\left(x'_i x'_j - \frac{1}{3}r'^2\delta_{ij}\right) = (*)$$

this is just the definition of Q

$\partial_t T^{00} = -\partial_i T^{0i}$ and integrate by parts again

$$(*) = \int d^3x' \, T^{00}(t, \bar{x}') \, \overset{\circ}{x}_K \left(\delta_{iK} x'_J + \delta_{JK} x'_i \right)$$

$$\Rightarrow \frac{dF_i}{dV} = -\frac{2}{5} T^{00}(t\, x) x_J \frac{d^5 Q_{iJ}}{dt^5}(t) \quad (3.109)$$

$$T^{00} = \rho \longrightarrow F = -\frac{2}{5} \frac{d^5 Q_{iJ}}{dt^5} \underbrace{\int d^3x' \rho(t, x') x'_J}$$

this is the location of the CENTER OF MASS times the mass itself $m x_J$

$$\longrightarrow \boxed{F_i = -\frac{2}{5} m x_J \frac{d^5 Q_{iJ}}{dt^5}} \quad (3.112)$$

FIRST_ORDER RADIATION REACTION

As GW are emitted, the source reacts as if with a force F_i

To summarize

$h \sim \ddot{Q}$ strain

$P \sim \dddot{Q}$ power

$F \sim \overset{(5)}{Q}$ radiation reaction