1. GW PROPAGATION

RECAP OF LINEARIZED GRAVITY

(assuming you have seen this already in a GR class)

MAGGIORE CHAP 1

$\boxed{\begin{array}{l} G=c=1 \\ \text{everywhere} \end{array}}$

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi T_{\mu\nu}$$

metric is flat + perturbation

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$$

If only spatial indexes, the flat metric is δ^{ij} so we don't care about upper and lower indexes

We raise/lower indexes with $g_{\mu\nu}$

$$\bar{h}_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h^{\sigma}{}_{\sigma}$$

$\partial^{\nu}\bar{h}_{\mu\nu}=0$ Lorentz gauge ← still leaves some freedom

Propagate as a wave!

$$\boxed{\Box\bar{h}_{\mu\nu}=-16\pi T_{\mu\nu}}$$

LINEARIZED EINSTEIN EQUATIONS (1.24)

conservation of energy and momentum $\longrightarrow$ $\partial^{\nu}T_{\mu\nu}=0$

TT gauge (transverse traceless) $\bar{h}^{\mu}{}_{\mu}=0$

$\longrightarrow h^{0\mu}=0 \quad h^{i}{}_{i}=0 \quad \partial^{j}h_{ij}=0$

Lambda tensor (TT projection)

TT gauge ← $h^{TT}_{ij}=\Lambda_{ijkl}\,h_{kl}$ → Lorentz gauge

$$\Lambda_{ijkl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl} \qquad (1.36)$$

$$P_{ij}(\hat{n}) = \delta_{ij} - n_i n_j$$

→ general expression of the TT projector

if S_{ij} symmetric then $S^{TT}_{ij} = \Lambda_{ijkl}S_{kl}$ also symmetric

plane wave propagating along z

$$h^{TT}_{ij}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos(\omega(t-z))$$

$h_+, h_\times$ POLARIZATIONS

ENERGY OF GWs

(Sec 1.4)

Do GWs carry energy? If yes, do they generate curvature? (in GR energy = curvature)

Do GWs deposit energy in the background spacetime?

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ not enough!

if the background is flat, we are preventing GWs to carry energy by construction

$$(1.106) \quad g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) \qquad |h_{\mu\nu}| \ll 1$$

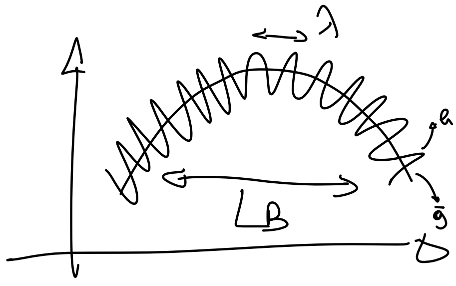
"background" "perturbation" → does this really propagate as a wave?

How to separate background and perturbation?

$\lambda \ll L_B$

wavelength of the perturbation

variation scale of the background



Alternatively $f_{GW} >> f_B$

frequency content of h — frequency content of $\bar{g}$

$$R_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\rho}_{\rho} \right)$$ Einstein's equation

Expand Ricci $R_{\mu\nu} = \bar{R}_{\mu\nu} + R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + \dots$ (1.110)

$\bar{R}_{\mu\nu}$: using $\bar{g}_{\mu\nu}$; $R^{(1)}_{\mu\nu}$: $O(h)$; $R^{(2)}_{\mu\nu}$: $O(h^2)$

→ $\bar{R}_{\mu\nu}$ only low frequency ($\bar{g}$)

→ $R^{(1)}_{\mu\nu}$ only high frequency (h)

→ $R^{(2)}_{\mu\nu}$ both low and high frequency (h and h term can interact)

Average over a length scale $\bar{\ell}$ with $\lambda << \bar{\ell} << L_B$
High frequency modes average out and the low frequency modes survive

$$\bar{R}_{\mu\nu} = - \langle R^{(2)}_{\mu\nu} \rangle + 8\pi \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle \qquad (1.122)$$

Now define $\bar{T}_{\mu\nu}$ and $t_{\mu\nu}$

$$\bar{T}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{T} \equiv \langle T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \rangle \qquad (1.123)$$

$$t_{\mu\nu} \equiv -\frac{1}{8\pi} \langle R^{(2)}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} R^{(2)} \rangle \qquad (1.125)$$

$$\Longrightarrow \boxed{\bar{R}_{\mu\nu} - \frac{1}{2} \bar{g}_{\mu\nu} \bar{R} = 8\pi \left(\bar{T}_{\mu\nu} + t_{\mu\nu} \right)} \qquad (1.130)$$

curvature of the background — energy-momentum of the background — energy-momentum of the GWs

Tedious calculation (Riemann tensor, Christoffel symbols, expand to $O(h^2)$, integrate by parts)

$$t_{\mu\nu} = \frac{1}{32\pi}\langle \partial_\mu h_{\alpha\beta}\, \partial_\nu h^{\alpha\beta} \rangle$$

(it's in the Lorentz gauge but one can verify it satisfies the TT condition as well

$$t^{00} = \frac{1}{32\pi}\langle \dot{h}^{TT}_{ij}\, \dot{h}^{TT}_{ij} \rangle = \frac{1}{16\pi}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \qquad (1.135\_136)$$

The RHS of (1.130) satisfies the Bianchi identity

$$\rightsquigarrow \nabla^\mu (T_{\mu\nu} + t_{\mu\nu}) = 0$$

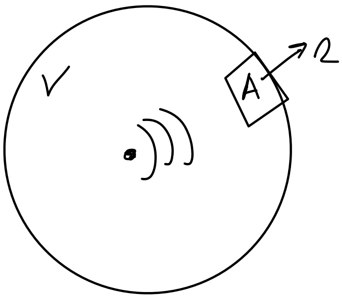
Far from the source $T_{\mu\nu} = 0$ and $\nabla^\mu \sim \partial^\mu$

$$\longrightarrow \boxed{\partial^\mu t_{\mu\nu} = 0}$$ stress_energy conservation of GWs

it's a conserved quantity so it makes sense to interpret it as a stress_energy tensor

ENERGY FLUX

$$\partial^\mu t_{\mu\nu} = 0 \longrightarrow \int_V d^3x \left(\partial_0 t^{00} + \partial_i t^{i0}\right) = 0$$



$E_V = \int d^3x\, t^{00}$ energy

$$\rightsquigarrow \frac{dE}{dt} = -\int d^3x\, \partial_i t^{0i} \overset{\text{stokes}}{=} -\int dA\, t^{0r}$$

$$t^{0r} = \frac{1}{32\pi}\langle \partial^0 h^{TT}_{ij}\, \partial^r h^{TT}_{ij} \rangle \qquad (1.146)$$

In general one has $h^{TT}_{ij}(t,r) = \underbrace{\frac{1}{r}}_{\text{amplitude}} f_{ij}(\underbrace{t - \frac{r}{c}}_{\text{retarded time}})$

(will prove this later, but it's like electromagnetism...)

$$\partial_r h^{TT}_{ij} = \frac{1}{r}\partial_r f_{ij}\left(t-\frac{r}{c}\right) + O\left(\frac{1}{r^2}\right)$$

but $\partial_r f_{ij}\left(t-\frac{r}{c}\right) = -\partial_t f_{ij}\left(t-\frac{r}{c}\right)$ — for a generic function that depends on $t-\frac{r}{c}$

$$\leadsto \partial^r h^{TT}_{ij} = \partial_r h^{TT}_{ij} = -\partial_0 h^{TT}_{ij} = +\partial^0 h^{TT}_{ij}$$

$$(1.135)+(1.146) \Longrightarrow t^{0r} = t^{00} \Longrightarrow \frac{dE}{dt} = -\int dA\, t^{00} \quad (1.151)$$

Flux decreasing, GW carry energy away from the source. The flux is

$$\frac{dE}{dA\,dt} = +t^{00} = \frac{1}{32\pi}\langle \dot{h}^{TT}_{ij}\dot{h}^{TT}_{ij}\rangle = \frac{1}{16\pi}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle \quad (1.153)$$

$$\leadsto \boxed{\frac{dE}{dt} = \frac{r^2}{32\pi}\int d\Omega \langle \dot{h}^{TT}_{ij}\dot{h}^{TT}_{ij}\rangle}$$

GW energy flux

$(dA = r^2 d\Omega)$ (recall $h \propto \frac{1}{r}$)

Similar calculation for the momentum

$$\frac{dp^k}{dt} = -\frac{r^2}{32\pi}\int d\Omega \langle \dot{h}^{TT}_{ij}\partial^k \dot{h}^{TT}_{ij}\rangle$$