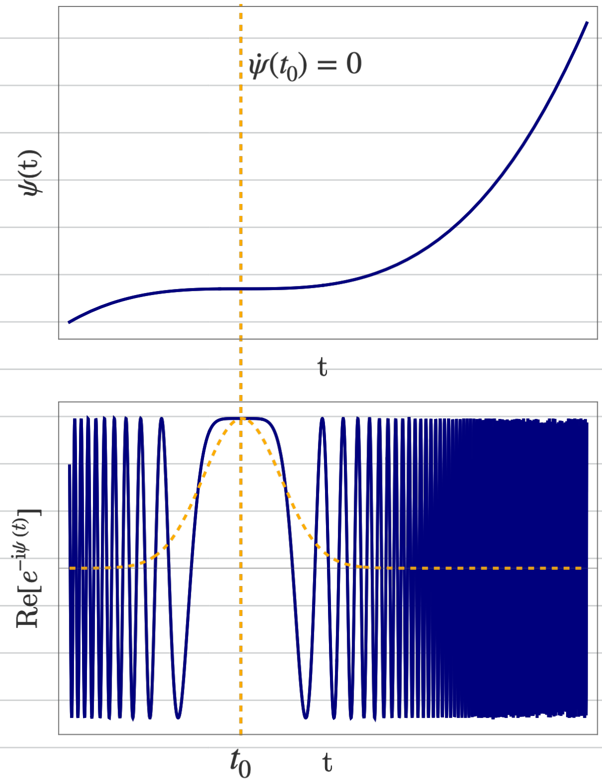
Stationary Phase Approximation (SPA)

Need Fourier transform for GW data analysis (match filtering, Parameter estimation)



SPA $\Rightarrow$ directly compute $\tilde{h}(f)$
when radiation-reaction time scale $\ll$ orbital period

what do we need to apply it ?

1. Amplitude varies slower than phase
2. orbital frequency / phase are posative and monotonically increasing

Outline

1. derive SPA for cosine time domain signal
2. Apply it to a circular Newtonian binary
3. discus how to extend the calculation

Resources: 0906.0313, SPA wikepedia, 2503.0496, Maggiore

$$f(t) = A(t)\, e^{-i\psi(t)}$$

$$I = \int f(t)\, dt$$

$$\psi(t) = \psi(t_0) + \dot{\psi}(t_0)(t-t_0) + \tfrac{1}{2}\ddot{\psi}(t_0)(t-t_0)^2 + \ldots$$

$$A(t) = A(t_0) + \ldots \qquad \text{(Amplitude varies slowly)}$$

$$I = A(t_0)\, e^{-i[\psi(t_0) + \pi/4]} \left[\frac{2\pi}{\ddot{\psi}(t_0)}\right]^{1/2}$$

$$h^{ij} = h_+(t)\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix} + h_\times(t)\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$\iota = 0$

$$h_+ = \tfrac{1}{2}\cancelto{1}{(1+\cos^2\iota)}\, A(t)\cos[2\phi(t)] \qquad h_\times = \cancelto{1}{\cos\iota}\, A(t)\cos[2\phi(t)]$$

$\phi(t)$ = orbital phase

$m = m_1 + m_2$

$r = |\vec{r}_1 - \vec{r}_2|$

$\eta = \dfrac{m_1 m_2}{m^2}$ $\quad m\eta = \mu$

$v = |\vec{v}_1 - \vec{v}_2|$

$$h^{ij} = \frac{1}{d}\ddot{I}^{ij} + \ldots \sim \frac{1}{d}\omega^2 m r^2$$

$$A(t) = -4\frac{1}{d}\eta m \omega^2 r^2 + \ldots$$

Circular binary $\Rightarrow$ relate r, ω, v

$$\left.\begin{aligned} \omega^2 &= \frac{m}{r^3} + \ldots \\ v &= \omega r \end{aligned}\right\} \quad x \equiv (m\omega)^{2/3} \quad \left\}\begin{aligned} \omega^2 &= \frac{x}{m^{2}}{}_3 \\ r &= \frac{m}{x} + \ldots \\ v^2 &= x + \ldots \end{aligned}\right.$$

$$A(t) = \boxed{-4\frac{1}{d}\eta m x + \ldots}$$

While we are at it,

$$E = \tfrac{1}{2}m\eta v^2 - \frac{m^2\eta}{r} = \boxed{-\tfrac{1}{2}m\eta x + \ldots}$$ total energy

$$P \sim \dddot{I}^{ij}\dddot{I}_{ij} + \ldots \sim \omega^6 (m r^2)^2$$
$$= \frac{32}{5}\eta^2 m^2 \omega^6 r^4 = \boxed{\frac{32}{5}\eta^2 x^5 + \ldots}$$ Power radiated by GWs

$$h(t) = h_+ - i h_\times = A e^{-2i\phi}$$

$$\tilde{h}(f) = \int h(t) e^{2\pi i f t} \, dt$$

$$= \int A(t) e^{2\pi i f t} e^{-2i\phi(t)} \, dt = \tilde{h}(f) = \tilde{A}(f) e^{-i\tilde{\phi}(f)}$$

$$\psi(t) = 2i\left[\pi f t - \phi(t)\right]$$

$$\dot{\psi}(t) = 2i\left[\pi f - \dot{\phi}(t)\right]$$

$$\ddot{\psi}(t) = -2i\ddot{\phi}(t) = -2i\dot{\omega}(t)$$

stationary point: $\dot{\psi}(t_0) = 0$

$$\dot{\phi}(t_0) = \pi f \equiv \omega(t_0)$$

$$\boxed{\tilde{A}(f) = A(t_0)\left(\frac{\pi}{\dot{\omega}(t_0)}\right)^{1/2}}$$

= we need to solve for these

$$\tilde{\phi}(f) = 2t_0\,\omega(t_0) - 2\phi(t_0) - \pi/4$$

$$x_0 = x(t_0) = \left[m\,\omega(t_0)\right]^{2/3} = (\pi m f)^{2/3}$$

$$\boxed{\tilde{\phi}(f) = 2t_0 \frac{x_0^{3/2}}{m} - 2\phi(t_0) - \pi/4}$$

Solve for $\tilde{\Phi}(f)$ First

Ballance equation $\quad P = -\frac{dE}{dt} = -\frac{dE}{dx}\frac{dx}{dt} = -E'\frac{dx}{dt}$

Not Newtonian! binary loose energy to GW

Find t_0:

$$\frac{E'}{P} = -\frac{5}{64}\frac{m}{\eta}x^{-5}$$

$$dt = -\frac{E'}{P}dx$$

$$t_0 = t_c - \int \frac{E'}{P}dx \Big|_{x=x_0}$$

$$t_c + \frac{5}{64}\frac{m}{\eta}\int x^{-5}dx \Big|_{x=x_0}$$

$$\boxed{t_0 = t_c - \frac{5}{256}\frac{m}{\eta}x_0^{-4}}$$

Find $\phi(t_0)$

$$\omega = \frac{d\phi}{dt} = \frac{d\phi}{dx}\frac{dx}{dt} = \underline{-\frac{d\phi}{dx}\frac{P'}{E} = \frac{x^{3/2}}{m}}$$

$$d\phi = -\frac{E'}{P}\frac{x^{3/2}}{m}dx$$

$$\phi(t_0) = \phi_c - \int \frac{E'}{P}\frac{x^{3/2}}{m}dx \Big|_{x=x_0}$$

$$= \phi_c + \frac{5}{64}\frac{1}{\eta}\int x^{-7/2}dx \Big|_{x=x_0}$$

$$\boxed{\phi(t_0) = \phi_c - \frac{1}{32}\frac{1}{\eta}x_0^{-5/2}}$$

$$\tilde{\Phi} = 2 t_0 \frac{x_0^{3/2}}{m} - 2\phi(t_0) - \pi/4$$

$$= 2\left(t_c - \frac{5}{256}\frac{m}{\eta} x_0^{-4}\right)\frac{x_0^{3/2}}{m} - 2\left(\phi_c - \frac{1}{32}\frac{1}{\eta} x_0^{-5/2}\right) - \pi/4$$

$$= 2 t_c \frac{x_0^{3/2}}{m} - 2\phi_c + \frac{3}{128}\frac{1}{\eta} x_0^{-5/2} - \pi/4$$

$$\tilde{\Phi}(f) = 2\pi f t_c - 2\phi_c + \frac{3}{128}\frac{1}{\eta}(\pi m f)^{-5/3}$$

$$\boxed{\mathcal{M} = m\eta^{3/5}}$$

chirp mass!

$$\boxed{\tilde{\Phi}(f) = 2\pi f t_c - 2\phi_c + \frac{3}{128}\mathcal{M}^{-5/3}(\pi f)^{-5/3}}$$

Next, Solve for $\tilde{A}(f)$

Find $\dot{\omega}(t_0)$:

$$\dot{\omega} = \frac{3}{2}\frac{1}{m} x^{1/2}\frac{dx}{dt}$$

$$\dot{\omega}(t_0) = \frac{3}{2}\frac{1}{m} x^{1/2}\left.\frac{P}{E'}\right|_{x = x_0}$$

$$\boxed{\dot{\omega}(t_0) = \frac{96}{5}\frac{\eta}{m^2} x_0^{11/2}}$$

Find $A(t_0)$:

$$\boxed{A(t_0) = -4\frac{1}{d}\eta m x_0}$$

$$\tilde{A}(f) = A(t_0)\left(\frac{\pi}{\dot{\omega}(t_0)}\right)^{1/2}$$

$$= -4\frac{1}{d}\eta m x_0 \pi^{1/2}\left(\frac{5}{96}\frac{m^2}{\eta} x_0^{-11/2}\right)^{1/2}$$

$$= -\left(\frac{5\pi}{6}\right)^{1/2}\frac{1}{d} m^2 \eta^{1/2} x^{-7/4}$$

$$\tilde{A}(f) = -\left(\frac{5\pi}{6}\right)^{1/2}\frac{1}{d} m^2 \eta^{1/3}(\pi m f)^{-7/6}$$

$$\boxed{\tilde{A}(f) = -\left(\frac{5\pi}{6}\right)^{1/2}\frac{1}{d}\mathcal{M}^{5/6}(\pi f)^{-7/6}}$$

Note: $\tilde{h}_+ = \frac{1}{2}\tilde{h}$, $\tilde{h}_\times = \frac{i}{2}h$ $\Rightarrow$ $\tilde{h} = \tilde{h}_+ - i\tilde{h}_\times$

How to extend calculation?

must add corrections to:

Keplers law $[\omega(r)]$, $A(t)$, $E(x)$, $P(x)$